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# REGULARITY FOR ENERGY-MINIMIZING AREA-PRESERVING DEFORMATIONS

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ABSTRACT. In this paper we establish the square integrability of the nonnegative hydrostatic pressure  $p$ , that emerges in the minimization problem

$$\inf_{\mathcal{K}} \int_{\Omega} |\nabla \mathbf{v}|^2, \quad \Omega \subset \mathbb{R}^2$$

as the Lagrange multiplier corresponding to the incompressibility constraint  $\det \nabla \mathbf{v} = 1$  a.e. in  $\Omega$ . Our method employs the Euler-Lagrange equation for the mollified Cauchy stress  $\mathbf{C}$  satisfied in the image domain  $\Omega^* = \mathbf{u}(\Omega)$ . This allows to construct a convex function  $\psi$ , defined in the image domain, such that the measure of the normal mapping of  $\psi$  controls the  $L^2$  norm of the pressure. As a by-product we conclude that  $\mathbf{u} \in C_{\text{loc}}^{\frac{1}{2}}(\Omega)$  if the dual pressure (introduced in [6]) is nonnegative.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^2$  and  $\mathcal{K} = \{\mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^2), \det \nabla \mathbf{v} = 1 \text{ a.e. in } \Omega\}$ . For  $\mathbf{v} \in \mathcal{K}$  we define the stored energy as

$$(1.1) \quad E[\mathbf{v}] = \int_{\Omega} |\nabla \mathbf{v}|^2, \quad \mathbf{v} \in \mathcal{K}.$$

Let us recall the definition of local minimizers [1], [2], [6].

**Definition 1.1.** We say that an area-preserving deformation  $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^2)$  is a *local minimizer* if for all area preserving (or incompressible) deformations  $\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2)$  with  $\text{supp}(\mathbf{w} - \mathbf{u}) \subset \Omega$  the following holds

$$(1.2) \quad \int_{\Omega} |\nabla \mathbf{u}|^2 \leq \int_{\Omega} |\nabla \mathbf{w}|^2.$$

Our primary interest is to analyze the properties of the local minimizers of  $E[\cdot]$  and the integrability of the hydrostatic pressure  $p$  sought as the Lagrange multiplier corresponding to the incompressibility constraint  $\det \nabla \mathbf{v} = 1$ . The sufficiently regular local minimizers solve the system

$$(1.3) \quad \begin{cases} \operatorname{div} \mathbf{T} = 0 & \text{in } \Omega, \\ \det \nabla \mathbf{u} = 1 & \text{a.e. in } \Omega, \end{cases}$$

where  $\mathbf{T} = \nabla \mathbf{u} + p(\nabla \mathbf{u})^{-t}$  is the first Piola-Kirchhoff tensor and  $(\nabla \mathbf{u})^{-t}$  is the transpose of the inverse matrix, see [7], pages 371 and 379. Since  $\det \nabla \mathbf{u} = 1$  we have

$$(1.4) \quad (\nabla \mathbf{u})^{-1} = \begin{pmatrix} u_2^2 & -u_2^1 \\ -u_1^2 & u_1^1 \end{pmatrix}, \quad (\nabla \mathbf{u})^{-t} = \begin{pmatrix} u_2^2 & -u_1^2 \\ -u_2^1 & u_1^1 \end{pmatrix}.$$

From (1.4) we deduce that (1.3) is equivalent to the system

$$(1.5) \quad \begin{cases} \operatorname{div}[\nabla u^1 - p \mathcal{J} \nabla u^2] = 0, \\ \operatorname{div}[\nabla u^2 + p \mathcal{J} \nabla u^1] = 0, \\ \det \nabla \mathbf{u} = 1. \end{cases}$$

Here  $\mathcal{J}$  is the  $90^\circ$  counterclockwise rotation

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$$(1.6) \quad \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For  $\mathbf{u} \in W^{1,2}(\Omega)$  the equations (1.3) or (1.5) cannot be justified. In fact the term  $p(\nabla \mathbf{u})^{-t}$  is not well-defined unless  $\nabla \mathbf{u}$  is better than  $L^2$  integrable, see [2]. The lack of higher integrability of  $\nabla \mathbf{u}$  produces a number of technical difficulties, see [6]. To circumvent them author and N. Chaudhuri succeeded to compute the first variation of the energy (1.6) in the image domain  $\Omega^* = \mathbf{u}(\Omega)$  under very weak assumptions (note that  $\mathbf{u}$  is open map [10]). For  $\mathbf{u} \in W^{s,l}(\Omega)$  with  $s > \frac{2}{l} + 1$  this was done in [8], Theorem 5.1. Below we formulate one of the main results from [2] relevant to the present work.

**Proposition 1.2.** *Let  $\mathbf{u} \in \mathcal{K}$  be a local minimizer of (1.1). Consider the matrix*

$$(1.7) \quad \sigma_{ij}(y) = \sum_m u_m^i(\mathbf{u}^{-1}(y)) u_m^j(\mathbf{u}^{-1}(y))$$

where  $y \in \mathbf{u}(\Omega) = \Omega^*$  and  $\mathbf{u}^{-1}$  is the inverse of  $\mathbf{u}$  ( $\mathbf{u}^{-1}$  is well-defined see Remark 3.3 [10]). If  $\rho_\varepsilon$  is a mollification kernel and  $\sigma^\varepsilon = \sigma * \rho_\varepsilon$  then there is a  $C^\infty$  function  $q^\varepsilon$  such that

$$(1.8) \quad \operatorname{div} \sigma^\varepsilon(y) + \nabla q^\varepsilon(y) = 0 \quad y \in \Omega^*,$$

The regularized equation (1.8) in the image domain plays the crucial role in the proof of Theorem A (see below), notably it links (1.3) to the Monge-Ampère equation and from there we infer that  $\{q^\varepsilon\}$  is uniformly bounded in  $L^2_{\text{loc}}(\Omega^*)$ .

**Theorem A.** *Let  $\mathbf{u} \in \mathcal{K}$  be a local minimizer of  $E[\cdot]$ . If there is a sequence of  $q^{\varepsilon_j} \geq 0$  solving (1.8) such that  $q^{\varepsilon_j}$  converges to a nonnegative Radon measure in  $B_1 \subset \Omega^*$ , then there is a convex function  $\psi^\varepsilon$  defined in  $B_1$  such that*

$$D^2 \psi^\varepsilon = \operatorname{adj} \sigma^\varepsilon + q^\varepsilon \mathbb{I}$$

where  $\operatorname{adj} \sigma^\varepsilon = (\sigma^\varepsilon)^{-1} \det \sigma^\varepsilon$  and  $\mathbb{I}$  is the identity matrix. Moreover,

- there is a subsequence  $q^{\varepsilon_{j(m)}}$  and  $q \in L^2_{\text{loc}}(\Omega^*)$  such that  $q^{\varepsilon_{j(m)}} \rightarrow q$  strongly in  $L^2_{\text{loc}}(\Omega^*)$ ,
- there is a convex function  $\psi : B_1 \mapsto \mathbb{R}$  such that  $\psi^{\varepsilon_{j(m)}} \rightarrow \psi$  uniformly on the compact subsets of  $B_1$ .

In [2] the authors found a representation for  $q^\varepsilon$  given by a sum of Calderón-Zygmund type singular integrals of  $\sigma_{ij}^\varepsilon(y)$ . As a result  $q^\varepsilon$  inherits the "half" of the integrability of  $\nabla \mathbf{u}$ . In other words  $\{q^\varepsilon\}$  is uniformly bounded in  $L^{1+\frac{\delta}{2}}_{\text{loc}}(\Omega^*)$  if  $\nabla u \in L^{2+\delta}(\Omega)$ ,  $\delta > 0$  and in  $L^1_{\text{loc}}(\Omega^*)$  if  $|\nabla \mathbf{u}|^2 \in L \log(2+L)(\Omega)$ . This observation gives rise to the following question: Does the higher integrability of the pressure  $q$  translate to  $\nabla \mathbf{u}$ ?

Theorem A gives a partial answer to this question: if  $B_1 \subset \Omega^*$ ,  $q \in L^{2+\delta}(B_1)$ ,  $\delta > 0$  and  $\sigma \in L^2(B_1)$  then it follows from Lemma 7.1 <sup>1°</sup> that  $D^2 \psi = \operatorname{adj} \sigma + q \mathbb{I}$  and  $D^2 \psi \in L^2(B_{\frac{7}{8}})$ . Since by (1.7)  $\sigma(y) = [\nabla \mathbf{u}(\nabla \mathbf{u})^t] \circ \mathbf{u}^{-1}(y)$ ,  $y \in \Omega^*$  we infer that  $\det \operatorname{adj} \sigma = 1$ , which is equivalent to the Monge-Ampère equation

$$\det [D^2 \psi - q \mathbb{I}] = 1$$

satisfied a.e. in  $B_1$ . Hence from the regularity theory available for the Monge-Ampère equation we will conclude higher integrability for  $D^2 \psi$  in  $B_{\frac{1}{2}}$ , which translates to  $\nabla \mathbf{u}$  in  $\Omega$  through the equation  $D^2 \psi = \operatorname{adj} \sigma + q \mathbb{I}$  and the inverse mapping theorem.

As one can observe from (1.8), the pressure  $q^\varepsilon$  is defined modulo a constant. The assumption  $q^{\varepsilon_j} \geq 0$  seems a natural one since from a purely physical point of view the pressure must be nonnegative. From Theorem A

we can conclude that the first equation in (1.3) is well defined in  $\Omega$ . Moreover applying the duality argument from [6] we infer that there is a function  $P : \Omega^* \mapsto \mathbb{R}$  such that the pair  $(\mathbf{u}^{-1}, P)$  is a solution the corresponding Euler-Lagrange equations in  $\Omega^*$ , see Theorem 2 [6]. Combining Theorem A with this observation we obtain

**Theorem B.** *Let  $\mathbf{u} : \Omega \mapsto \mathbb{R}^n$  and  $q \in L^2(\Omega^*)$  be as in Theorem A.*

- 1° *Then  $p(x) = q(\mathbf{u}(x))$ ,  $x \in \Omega$  is locally  $L^2$  integrable in  $\Omega$ ,  $p(x)(\nabla \mathbf{u})^{-t} \in L^2_{loc}(\Omega)$  and the pair  $(\mathbf{u}, p)$  solves the equation*

$$\operatorname{div}[\nabla \mathbf{u} + p(\nabla \mathbf{u})^{-t}] = 0 \quad \text{in } \Omega$$

*in the weak sense.*

- 2° *Let  $\mathbf{v} = \mathbf{u}^{-1}$  and  $Q$  be the dual pressure in  $\Omega$  corresponding to  $\mathbf{v}$ ,  $Q(\mathbf{v}(z)) = P(z)$ . If  $Q \geq 0$  then  $\mathbf{u} \in C^{\frac{1}{2}}_{loc}(\Omega)$ .*

The paper is organized as follows: Section 2 is devoted to the construction of the family of functions  $\psi^\varepsilon$ . Then we prove uniform estimates for this family using some geometric ideas and the Poincaré-Wirtinger's theorem for the functions of bounded variation (or  $BV$ -functions, see [4]). This is contained in Section 3. A lower estimate for the  $\det \operatorname{adj} \sigma^\varepsilon$  is established in Section 4. Next, in order to prove Theorem A, we recall the notion of generalized solution of the Monge-Ampère equation and define the corresponding normal mapping in Section 5. The proof of Theorem A is given in Section 6. Section 7 contains a brief discussion of the properties of the convex function  $\psi$  and its Legendre-Fenchel transformation. Finally, Section 8 contains the proof of Theorem B.

## 2. THE EULER-LAGRANGE EQUATION IN IMAGE DOMAIN

In this section we construct a convex function  $\psi^\varepsilon$  such that the mollification of the Cauchy stress tensor  $\mathbf{C}_{ij} = \sigma_{ij} + q\delta_{ij}$  is the Hessian of  $\psi^\varepsilon$ .

We start by recalling that if  $\mathbf{w}$  is  $C^\infty$  divergence free vectorfield in 2D then there is a scalar  $C^\infty$  function  $\varphi$  such that  $\mathbf{w} = \mathcal{J} D\varphi = (-D_2\varphi, D_1\varphi)$ .

Suppose that  $B_1 \subset \Omega^*$ . From the mollified equation (1.8) it follows that the vectorfields  $(\sigma_{11}^\varepsilon + q^\varepsilon, \sigma_{12}^\varepsilon)$  and  $(\sigma_{21}^\varepsilon, \sigma_{22}^\varepsilon + q^\varepsilon)$  are divergence free in  $\Omega^*$ . Hence there are two scalar functions  $\varphi_1^\varepsilon, \varphi_2^\varepsilon$  such that  $\varphi_i^\varepsilon \in C^\infty(B_1)$ ,  $i = 1, 2$  and

$$(2.1) \quad \begin{aligned} (\sigma_{11}^\varepsilon + q^\varepsilon, \sigma_{12}^\varepsilon) &= \mathcal{J} D\varphi_1^\varepsilon = (-\partial_2\varphi_1^\varepsilon, \partial_1\varphi_1^\varepsilon), \\ (\sigma_{21}^\varepsilon, \sigma_{22}^\varepsilon + q^\varepsilon) &= \mathcal{J} D\varphi_2^\varepsilon = (-\partial_2\varphi_2^\varepsilon, \partial_1\varphi_2^\varepsilon). \end{aligned}$$

Since

$$(2.2) \quad [\sigma_{ij}(z)] = \begin{pmatrix} |\nabla u^1(\mathbf{u}^{-1}(z))|^2 & \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) \\ \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) & |\nabla u^2(\mathbf{u}^{-1}(z))|^2 \end{pmatrix}$$

and  $\sigma_{ij}^\varepsilon = \sigma_{ij} * \rho_\varepsilon$ , where  $\rho_\varepsilon$  is a mollifying kernel, we conclude that  $\sigma_{ij}^\varepsilon$  is symmetric. Moreover the gradient matrix of the mapping  $\Phi^\varepsilon = (\varphi_1^\varepsilon, \varphi_2^\varepsilon)$  is

$$(2.3) \quad \nabla \Phi^\varepsilon = \begin{pmatrix} \partial_1\varphi_1^\varepsilon & \partial_2\varphi_1^\varepsilon \\ \partial_1\varphi_2^\varepsilon & \partial_2\varphi_2^\varepsilon \end{pmatrix} = \begin{pmatrix} \sigma_{12}^\varepsilon & -\sigma_{11}^\varepsilon - q^\varepsilon \\ \sigma_{22}^\varepsilon + q^\varepsilon & -\sigma_{21}^\varepsilon \end{pmatrix}.$$

Therefore the mapping  $\Phi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon)$  is divergence free, because

$$\operatorname{div} \Phi^\varepsilon = \partial_1\varphi_1^\varepsilon + \partial_2\varphi_2^\varepsilon = \sigma_{12}^\varepsilon - \sigma_{21}^\varepsilon = 0$$

and the matrix  $\sigma_{ij}^\varepsilon$  is symmetric.

Thus, there is a scalar function  $\psi^\varepsilon$  such that  $\Phi^\varepsilon = \mathcal{J} \nabla \psi^\varepsilon$ . In other words  $\varphi_1^\varepsilon = -\partial_2 \psi^\varepsilon, \varphi_2^\varepsilon = \partial_1 \psi^\varepsilon$ , which in view of (2.1) implies the following identity for the Hessian of  $\psi^\varepsilon$

$$(2.4) \quad D^2 \psi^\varepsilon(y) = \begin{pmatrix} \sigma_{22}^\varepsilon(y) + q^\varepsilon(y) & -\sigma_{21}^\varepsilon(y) \\ -\sigma_{21}^\varepsilon(y) & \sigma_{11}^\varepsilon(y) + q^\varepsilon(y) \end{pmatrix}.$$

Furthermore,  $\det D^2 \psi^\varepsilon = \det \operatorname{adj} \sigma^\varepsilon + (q^\varepsilon)^2 + q^\varepsilon \operatorname{Tr} \sigma^\varepsilon$  and  $\det(D^2 \psi - q^\varepsilon \mathbb{I}) = \det \operatorname{adj} \sigma^\varepsilon$ , where  $\mathbb{I} = \delta_{ij}$  is the identity matrix.

**Lemma 2.1.** *If  $q^\varepsilon \geq C$  for some  $C \in \mathbb{R}$ , independent of  $\varepsilon$ , then  $\psi^\varepsilon(y) - \frac{C}{2}|y|^2$  are convex for any  $\varepsilon > 0$ .*

**Proof:** Let  $e = (a, b) \in \mathbb{R}^2$  and  $\partial_e = a\partial_1 + b\partial_2$ . Then using (2.2) and (2.4) we conclude

$$\begin{aligned} \partial_{ee} \psi^\varepsilon(z) &= a^2 \partial_{11} \psi^\varepsilon + 2ab \partial_{12} \psi^\varepsilon + b^2 \partial_{22} \psi^\varepsilon \\ &= a^2 \sigma_{22}^\varepsilon + 2ab \sigma_{12}^\varepsilon + b^2 \sigma_{11}^\varepsilon + q^\varepsilon(z)(a^2 + b^2) \\ &= |a \nabla_x u^2(\mathbf{u}^{-1}(z)) + b \nabla_x u^1(\mathbf{u}^{-1}(z))|^2 + q^\varepsilon(z)(a^2 + b^2) \\ &\geq C(a^2 + b^2). \end{aligned}$$

Therefore  $\psi(z) - \frac{C}{2}|z|^2$  is convex.  $\square$

**Remark 2.2.** *The pressure  $q^\varepsilon(z)$  is defined modulo a constant as it is seen from the equation (1.8). In particular,  $\psi^\varepsilon$  is determined modulo a quadratic polynomial. Thus if  $q_0^\varepsilon(z) = q^\varepsilon(z) - C$  then  $\psi_0^\varepsilon(z) = \psi^\varepsilon(z) - \frac{C}{2}|z|^2$  solves  $\det(D^2 \psi_0^\varepsilon - q_0^\varepsilon(z) \mathbb{I}) = \det \operatorname{adj} \sigma^\varepsilon$  and (2.4) holds with  $\psi^\varepsilon$  and  $q^\varepsilon$  replaced by  $\psi_0^\varepsilon$  and  $q_0^\varepsilon$  respectively.*

### 3. UNIFORM ESTIMATES FOR $\psi^\varepsilon$

**Lemma 3.1.** *Suppose that the sequence  $q^\varepsilon$  converges to a nonnegative Radon measure  $q$ . Then there is a positive constant  $C$  such that  $\sup_{\partial B_1} |\psi^\varepsilon| \leq C$ .*

**Proof:** By Helmholtz-Weyl decomposition [3],  $\Phi^\varepsilon = Dh^\varepsilon + \mathcal{J} D\eta^\varepsilon$  where  $h^\varepsilon$  solves the Neumann problem

$$(3.1) \quad \begin{cases} \Delta h^\varepsilon = 0 & \text{in } B_1, \\ Dh^\varepsilon \cdot \nu = \Phi^\varepsilon \cdot \nu & \text{on } \partial B_1. \end{cases}$$

Moreover  $-\Delta \eta^\varepsilon = \operatorname{curl} \Phi^\varepsilon = \sigma_{11}^\varepsilon + \sigma_{22}^\varepsilon + 2q^\varepsilon$  and  $\eta^\varepsilon = 0$  on  $\partial B_1$ .

By Poincaré-Wirtinger's theorem  $\tilde{\Phi}^\varepsilon = \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon \in BV(B_1, \mathbb{R}^2)$ , i.e.  $\varphi_i^\varepsilon - \int_{B_1} \varphi_i^\varepsilon \in BV(B_1), i = 1, 2$ . Since  $\Phi^\varepsilon$  is defined modulo a constant (see (2.3)), in what follows, we take  $\tilde{\Phi}^\varepsilon = \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon$ . Thus the estimate

$$(3.2) \quad \|\tilde{\Phi}^\varepsilon\|_{L^1(B_1)} = \left\| \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon \right\|_{L^1(B_1)} \leq C \sup \left\{ \left| \int_{B_1} \Phi^\varepsilon \operatorname{div} \xi \right|, \forall \xi \in C_0^1(B_1, \mathbb{R}^2), |\xi| \leq 1 \right\}$$

is true, with  $C > 0$  independent from  $\varepsilon$ .

On the other hand after integration by parts we get

$$(3.3) \quad \int_{B_1} \tilde{\Phi}^\varepsilon \operatorname{div} \xi = \int_{B_1} \Phi^\varepsilon \operatorname{div} \xi = - \int_{B_1} \xi \nabla \Phi^\varepsilon$$

for any  $\xi \in C_0^1(B_1, \mathbb{R}^2)$  which in conjunction with (2.3) gives

$$\begin{aligned}
(3.4) \quad \left| \int_{B_1} \varphi_1^\varepsilon \operatorname{div} \xi \right| &= \left| - \int_{B_1} \xi D \varphi_1^\varepsilon \right| \\
&= \left| \int_{B_1} \xi^1 \sigma_{12}^\varepsilon - \xi^2 (\sigma_{11}^\varepsilon + q^\varepsilon) \right| \\
&\leq \int_{B_1} [|\sigma_{11}^\varepsilon| + |\sigma_{12}^\varepsilon| + q^\varepsilon].
\end{aligned}$$

Similarly, one can check that  $\left| \int_{B_1} \varphi_2^\varepsilon \operatorname{div} \xi \right| \leq \int_{B_1} [|\sigma_{12}^\varepsilon| + |\sigma_{22}^\varepsilon| + q^\varepsilon]$ . Because  $\sigma_{ij} \in L^1$  and  $q^\varepsilon$  converges to a nonnegative Radon measure it follows that

$$\|\tilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C (\|\sigma_{ij}\|_{L^1(B_1)} + \|q\|_{\mathcal{M}(B_1)}),$$

where  $\mathcal{M}(B_1)$  is the space of measures in  $B_1$ .

Using Theorems 2.10 and 2.11 from [4] we conclude that the trace  $\Phi_0^\varepsilon \in L^1(\partial B_1)$  of  $\tilde{\Phi}^\varepsilon$  is well-defined and satisfies the following uniform estimate

$$(3.5) \quad \|\tilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)} \leq C \|\tilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C (\|\sigma_{ij}\|_{L^1(B_1)} + \|q\|_{\mathcal{M}(B_1)}).$$

In particular (3.5) implies that the Neumann problem (3.1) for  $h^\varepsilon$  is well-defined.

Next we have that  $\Phi^\varepsilon = \mathcal{J} \nabla \psi^\varepsilon = \nabla h^\varepsilon + \mathcal{J} \nabla \eta^\varepsilon$  or equivalently

$$\nabla \psi^\varepsilon - \nabla \eta^\varepsilon = -\mathcal{J} \nabla h^\varepsilon.$$

In particular  $\psi^\varepsilon - \eta^\varepsilon$  is harmonic in  $B_1$ . We want to estimate the tangential component of  $\nabla \psi^\varepsilon$  on the boundary  $\partial B_1$ . Let  $\tau$  be a unit tangent vector to  $\partial B_1$ , then

$$\nabla \psi^\varepsilon \cdot \tau = \nabla \eta^\varepsilon \cdot \tau - \mathcal{J} \nabla h^\varepsilon \cdot \tau = \nabla h^\varepsilon \cdot \nu,$$

where  $\nu = \mathcal{J} \tau$  is a unit vector normal to  $\partial B_1$ . Using polar coordinates  $(r, \theta)$ ,  $\theta \in (0, 2\pi)$ , we obtain that

$$(3.6) \quad \psi^\varepsilon(\theta) = \psi^\varepsilon(0) + \int_0^\theta \nabla h \cdot \nu d\theta = \psi^\varepsilon(0) + \int_0^\theta \Phi_0^\varepsilon \cdot \nu d\theta.$$

Without loss of generality we assume that  $\psi^\varepsilon(0) = 0$  (see Remark 2.2). Thus

$$|\psi^\varepsilon(\theta)| \leq C \|\Phi_0^\varepsilon\|_{L^1(\partial B_1)}, \quad \forall \theta \in (0, 2\pi).$$

The desired result now follows from (3.5). □

**Lemma 3.2.** *Retain the assumptions of previous lemma. Then there is a constant  $C$ , such that  $\inf_{B_1} \psi^\varepsilon \geq C$  uniformly in  $\varepsilon$ .*

**Proof:** It suffices to prove that  $\nabla \psi^\varepsilon \in L^1(\partial B_1)$  uniformly in  $\varepsilon$ . Indeed,  $\psi^\varepsilon$  is convex hence if  $\psi^\varepsilon$  tends to  $-\infty$  then the  $\nabla \psi^\varepsilon$  becomes uniformly large on  $\partial B_1$ .

From Lemma 3.5 we have that

$$\nabla \psi^\varepsilon = \nabla \eta^\varepsilon - \mathcal{J} \nabla h^\varepsilon = \mathcal{J} (-\mathcal{J} \nabla \eta^\varepsilon - \nabla h^\varepsilon) = -\mathcal{J} \tilde{\Phi}^\varepsilon$$

implying the estimate

$$\|\nabla \psi^\varepsilon\|_{L^1(\partial B_1)} \leq \|\tilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)}.$$

The proof now follows if we recall (3.5). □

4. LOWER ESTIMATE FOR  $\det(\operatorname{adj} \sigma^\varepsilon)$ 

**Lemma 4.1.** *Let  $\sigma^\varepsilon = \sigma * \rho_\varepsilon$ , where  $\sigma(z) = [\nabla \mathbf{u}(\nabla \mathbf{u})^t] \circ \mathbf{u}^{-1}(z)$ ,  $z \in \Omega^\star$  then for any  $\varepsilon > 0$*

$$\det(\operatorname{adj} \sigma^\varepsilon(z)) \geq 1 \quad z \in \Omega^\star.$$

**Proof:** Using the definition of  $\sigma^\varepsilon(z)$  and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \det(\operatorname{adj} \sigma^\varepsilon) &= \sigma_{11}^\varepsilon \sigma_{22}^\varepsilon - \sigma_{12}^\varepsilon \sigma_{21}^\varepsilon \\ &= \int_{B_1} \sigma_{11} \rho_\varepsilon \int_{B_1} \sigma_{22} \rho_\varepsilon - \left( \int_{B_1} \sigma_{12} \rho_\varepsilon \right)^2 \\ &\geq \left( \int_{B_1} \sqrt{\sigma_{11} \sigma_{22}} \rho_\varepsilon \right)^2 - \left( \int_{B_1} \sigma_{12} \rho_\varepsilon \right)^2 \\ &= \int_{B_1} (\sqrt{\sigma_{11} \sigma_{22}} - \sigma_{12}) \rho_\varepsilon \int_{B_1} (\sqrt{\sigma_{11} \sigma_{22}} + \sigma_{12}) \rho_\varepsilon. \end{aligned}$$

By definition we have  $\sigma_{11} = |\nabla u^1|^2$ ,  $\sigma_{22} = |\nabla u^2|^2$  and  $\sigma_{12} = \sigma_{21} = \nabla u^1 \cdot \nabla u^2$ . Let  $\alpha$  be the angle between  $\nabla u^1$  and  $\nabla u^2$ . Recall that  $\det \nabla \mathbf{u} = |\nabla u^1| |\nabla u^2| \sin \alpha = 1$ . Then

$$\sqrt{\sigma_{11} \sigma_{22}} - \sigma_{12} = |\nabla u^1| |\nabla u^2| (1 - \cos \alpha) = |\nabla u^1| |\nabla u^2| 2 \sin^2 \frac{\alpha}{2} = \tan \frac{\alpha}{2}$$

and similarly have that

$$\sqrt{\sigma_{11} \sigma_{22}} + \sigma_{12} = |\nabla u^1| |\nabla u^2| (1 + \cos \alpha) = |\nabla u^1| |\nabla u^2| 2 \cos^2 \frac{\alpha}{2} = \cot \frac{\alpha}{2}.$$

Applying the Cauchy-Schwarz inequality one more time we obtain

$$\det(\operatorname{adj} \sigma^\varepsilon) \geq 1.$$

□

5. NORMAL MAPPING OF THE CONVEX FUNCTION  $\psi^\varepsilon$ 

In this section we will employ some basic concepts from the theory of generalized solutions of Monge-Ampère equation. Our notation follow that of the paper [11]. Let  $\psi$  be a convex function defined in  $B_1 \subset \mathbb{R}^2$ . For  $x \in B_1$  we let

$$\chi_\psi(x) = \{\xi \in \mathbb{R}^2 : \psi(y) \geq \psi(x) + \xi \cdot (y - x) \quad \forall y \in B_1\}.$$

For a set  $E \subset B_1$  we define the mapping

$$(5.1) \quad \chi_\psi(E) = \bigcup_{x \in E} \chi_\psi(x).$$

$\chi_\psi$  is called the normal mapping of  $\psi$ . For smooth convex  $\psi$ ,  $\chi_\psi$  coincides with the gradient mapping of  $\psi$ .

Let

$$\mathcal{C} = \{E \subset B_1 : \chi_\psi(E) \text{ is Lebesgue measurable}\}.$$

Then  $\mathcal{C}$  is a  $\sigma$ -algebra containing the Borel subsets of  $B_1$ , see [11]. For each  $E \in \mathcal{C}$  we define the set function

$$\omega(E) = |\chi_\psi(E)|$$

i.e. the Lebesgue measure of the normal mapping of  $E$ . It is easy to verify that for  $\psi \in C^2(B_1)$  we have

$$\omega(E) = \int_E \det D^2 \psi, \quad \text{for all Borel } E \in B_1.$$

It follows from Aleksandrov's theorem, see [11], that

$$|\{\xi \in \mathbb{R}^2 : \xi \in \chi_\psi(x) \cap \chi_\psi(y), \text{ for } x \neq y, x, y \in B_1\}| = 0.$$

As a consequence, we get that  $\omega$  is countably additive Radon measure.

Moreover, we have weak convergence for measure  $\omega$ . Indeed, let  $\psi_j$  be a sequence of convex functions and  $\psi_j \rightarrow \psi$  uniformly on compact subsets of  $B_1$ . Let  $\omega_j$  and  $\omega$  be the Radon measures associated with  $\psi_j$  and  $\psi$  respectively. Then  $\omega_j$  converges weakly on  $B_1$  to  $\omega$  in the space of measures  $\mathcal{M}(B_1)$  [11], i.e.

$$(5.2) \quad \limsup_{j \rightarrow \infty} \omega_j(K) \leq \omega(K)$$

for any compact set  $K \subset B_1$ , and

$$(5.3) \quad \liminf_{j \rightarrow \infty} \omega_j(U) \geq \omega(U)$$

for any open set  $U \subset B_1$ .

## 6. PROOF OF THEOREM A

Let  $\omega_j$  be the Radon measure corresponding to  $\psi^{\varepsilon_j}$ , for some sequence  $\{\varepsilon_j\}$ . By Lemmas 3.1 and 3.2 the sequence of convex functions  $\{\psi^{\varepsilon_j}\}$  is uniformly bounded in  $B_1$ . Thus for a subsequence, again denoted by  $\{\psi^{\varepsilon_j}\}$  we have  $\psi^{\varepsilon_j} \rightarrow \psi$  uniformly on the compact subsets of  $B_1$ . Clearly  $\psi$  is convex. Let  $\omega$  be the Radon measure corresponding to  $\psi$ . By Lemma 4.1 we have that

$$\begin{aligned} (6.1) \quad \omega_j(B_r(x_0)) &= \int_{B_r(x_0)} \det D^2 \psi^{\varepsilon_j} \\ &= \int_{B_r(x_0)} \det(\text{adj } \sigma^{\varepsilon_j}(z)) + q^{\varepsilon_j}(z) [|\nabla \mathbf{u}(\mathbf{u}^{-1}(z))|^2 * \rho_{\varepsilon_j}] + (q^{\varepsilon_j}(z))^2 dz \\ &\geq |B_r(x_0)| + \int_{B_r(x_0)} (q^{\varepsilon_j}(z))^2 dz \end{aligned}$$

for any open ball  $B_r(x_0) \subset B_1$ .

Now utilizing the weak convergence of the measures  $\omega_j \rightarrow \omega$  and (5.2) we obtain the following uniform

$$\int_K (q^{\varepsilon_j}(z))^2 dz \leq C + \omega(K)$$

for any compact set  $K \subset B_1$ . Then a customary compactness argument in  $L^2$  finishes the proof.  $\square$

## 7. PROPERTIES OF $\psi$

The convex function  $\psi$  enjoys a number of remarkable properties which are summarized in the following

**Lemma 7.1.** *Let  $\psi$  be as in Theorem A. Then*

- 1°  $\psi$  is strictly convex and  $\psi \in W_{\text{loc}}^{2,1}(B_1)$ ,
- 2°  $\psi^* \in C^{1,1}$  where  $\psi^*$  is the Legendre-Fenchel transformation of  $\psi$  in  $B_{\frac{1}{2}}$ .

**Proof:** 1° Recall that  $q^\varepsilon$  is defined modulo a constant summand, see Remark 2.2. Thus, without loss of generality, we assume that  $q^\varepsilon \geq 1$ . Let  $y_0$  be an arbitrary point in  $B_1$ , then by Lemma 4.1  $\det D^2 \psi^\varepsilon \geq (q^\varepsilon)^2 \geq 1$ . Thus we conclude that

$$\omega_j(U) \geq |U|, \quad \forall \text{ open } U \subset B_1.$$



Since  $\omega_j \rightharpoonup \omega$  weakly and in view of (5.3) the above inequality implies

$$\omega(U) \geq |U|.$$

Now the strict convexity of  $\psi$  follows from Aleksandrov's theorem, see [9], Chapter 2.3 Theorem 2.

The mollified matrices  $\sigma_{km}^{\varepsilon_j} \rightarrow \sigma_{km}$  strongly in  $L^1_{\text{loc}}(B_1)$  as  $\varepsilon_j \downarrow 0$  and  $q^{\varepsilon_j} \rightarrow q$  in  $L^2_{\text{loc}}$  at least for a subsequence. Moreover  $\{\psi^{\varepsilon_j}\}$  is uniformly bounded thanks to Lemmas 3.1 and 3.2, hence for a suitable subsequence  $\psi^{\varepsilon_j}$  will uniformly converge to a convex function  $\psi$  in any compact subset of  $B_1$ . Let us show that  $D^2\psi = \text{adj}\sigma + q\mathbb{I}$  a.e in  $B_1$ .

Indeed, let  $\eta \in C_0^\infty(B_1)$  and compute

$$\begin{aligned} \int \partial_k \psi \partial_i \eta &= \int \partial_k \psi^{\varepsilon_j} \partial_i \eta + o(1) \\ &= - \int \partial_{ik} \psi^{\varepsilon_j} \eta + o(1) \\ &= - \int [(\text{adj}\sigma^{\varepsilon_j})_{ik} + q^{\varepsilon_j} \delta_{ik}] \eta + o(1) \\ &\longrightarrow - \int [(\text{adj}\sigma)_{ik} + q \delta_{ik}] \eta. \end{aligned}$$

Hence  $\psi$  has generalized second order derivatives in  $L^1_{\text{loc}}(B_1)$  and  $D^2\psi = \text{adj}\sigma + q\mathbb{I}$  a.e in  $B_1$ .

**2°** Recall that the Legendre-Fenchel transformation  $\psi^*$  of  $\psi$  in  $B_{\frac{1}{2}}$  is given by

$$\psi^*(z) = \sup_{y \in B_{\frac{1}{2}}} (z \cdot y - \psi(y)), \quad z \in \chi_\psi(B_{\frac{1}{2}}).$$

Notice that by part **1°**  $\psi$  is strictly convex, hence it can be shown that  $\psi^*$  is  $C^1$  in the domain of  $\psi^*$ , see Chapter D of [5].

Let us denote  $B = B_{\frac{1}{2}}$  and  $B^* = \chi_\psi(B)$  where  $\chi_\psi$  is the normal mapping of  $\psi$ . Notice that  $B^*$  is bounded because  $\psi \in C^{0,1}(\overline{B_{\frac{1}{2}}})$ . Denote  $(B^\varepsilon)^* = \chi_{\psi^\varepsilon}(B)$ , then  $(\psi^\varepsilon)^*(z), z \in (B^\varepsilon)^*$  is smooth because  $\psi^\varepsilon$  is  $C^\infty$ . Furthermore from (2.4) we obtain

$$D^2(\psi^\varepsilon)^* = [D^2\psi^\varepsilon]^{-1} = \frac{1}{\det D^2\psi^\varepsilon} (\sigma^\varepsilon + q\mathbb{I})$$

or equivalently

$$\begin{aligned} \partial_{ij}(\psi^\varepsilon)^* &= \frac{\sigma_{ij}^\varepsilon + q\delta_{ij}}{\det \text{adj}\sigma + q^\varepsilon \text{Tr}\sigma^\varepsilon + (q^\varepsilon)^2} \\ &\leq \frac{1}{q^\varepsilon} \frac{\sigma_{ij}^\varepsilon + q\delta_{ij}}{\frac{1}{q^\varepsilon} + \text{Tr}\sigma^\varepsilon + q^\varepsilon} \\ &\leq \frac{1}{q^\varepsilon} \leq 1, \quad i = j \end{aligned}$$

if we assume that  $q^\varepsilon \geq 1$ , see Remark 2.2.

As for  $i \neq j$ , we use Lemma 4.1 to conclude

$$|\sigma_{12}^\varepsilon| \leq \sqrt{\sigma_{11}^\varepsilon \sigma_{22}^\varepsilon - 1} \leq \sqrt{\sigma_{11}^\varepsilon \sigma_{22}^\varepsilon} + 1 \leq \frac{\sigma_{11}^\varepsilon + \sigma_{22}^\varepsilon}{2} + 1.$$

Thus  $|D^2(\psi^\varepsilon)^*| \leq C$  uniformly in  $\varepsilon$ .

Next, we extend  $(\psi^\varepsilon)^*$  to  $B_R$  by the formula  $\sup_{z \in B_R} (y \cdot z - \psi^\varepsilon(y))$  with  $z \in B_R$  and  $R = \sup_{\varepsilon} \|\nabla \psi^\varepsilon\|_{L^\infty(B_{\frac{1}{2}})}$ . Thus in  $B_R$  we have a sequence of convex functions  $(\psi^\varepsilon)^*$  with uniformly bounded Hessian matrices. By a

customary compactness argument we can show that for at least a subsequence we have  $(\psi^{\varepsilon_j})^* \rightarrow \bar{\psi}$  for some convex function  $\bar{\psi}$ . It remains to show that  $\psi^* = \bar{\psi}$  in  $B^*$ .

From the definition of  $(\psi^\varepsilon)^*$  we have that  $(\psi^\varepsilon)^*(z) + \psi^\varepsilon(y) \geq y \cdot z$  and after passing to the limit we obtain  $\bar{\psi}(z) + \psi(y) \geq y \cdot z$  implying that  $\bar{\psi}(z) \geq \psi^*(z)$ . To get the converse inequality we use the uniform convergence

$$\bar{\psi}(z) \leftarrow (\psi^\varepsilon)^*(z) = \sup_{y \in B} (y \cdot z - \psi^\varepsilon(y)) \leq \sup_{y \in B} (y \cdot z - \psi(y)) + \sup_{y \in B} |\psi(y) - \psi^\varepsilon(y)| \rightarrow \psi^*(z).$$

This completes the proof.  $\square$

**Remark 7.2.** At each point  $z \in \text{int}B^*$ ,  $B^* = \chi_\psi(B_{\frac{1}{2}})$  we can define the lower Gauss curvature [9]

$$\underline{\omega}^*(z_0) = \liminf_{r \rightarrow 0} \frac{|\chi_{\psi^*}(B_r(z_0))|}{|B_r(z_0)|}.$$

If there is a constant  $m > 0$  such that  $\underline{\omega}^*(z_0) \geq m > 0$  for a.e.  $z_0 \in B^*$  then  $\sigma$  and  $q$  are bounded in  $B_{\frac{1}{2}}$ . In particular this will imply that  $\mathbf{u}$  is Lipschitz in  $\mathbf{u}^{-1}(B_{\frac{1}{2}}) \subset \Omega$ .

## 8. PROOF OF THEOREM B

The part  $1^\circ$  follows from change of variable formula [10] and Theorem A. To prove part  $2^\circ$  we employ the duality principle of  $\mathbf{u}$  and its inverse  $\mathbf{v} = \mathbf{u}^{-1}$  in [6], i.e.  $\mathbf{v}$  is a local minimizer of the dual problem in the image domain  $\Omega^* = \mathbf{u}(\Omega)$ . Hence we can apply Theorem A to the pair  $(\mathbf{v}, P)$  where  $\mathbf{v} = \mathbf{u}^{-1}$ . Thus, there is a convex function  $\eta^\varepsilon$  such that  $D^2\eta^\varepsilon = \text{adj}\tilde{\sigma}^\varepsilon + Q^\varepsilon \mathbb{I}$  where

$$\tilde{\sigma}_{ij}(z) = \sum_m v_m^i(\mathbf{v}^{-1}(z)) v_m^j(\mathbf{v}^{-1}(z)), \quad z \in \Omega$$

and  $\tilde{\sigma}^\varepsilon = \tilde{\sigma} * \rho_\varepsilon$  and  $Q^\varepsilon$  are the mollifications of  $\tilde{\sigma}$  and  $Q$  respectively. Note that  $Q(\mathbf{v}(z)) = P(z)$ ,  $z \in \Omega$ . In particular, for any  $B_r(x_0) \subset B_1 \subset \Omega$  we have

$$\begin{aligned} \int_{B_r(x_0)} |\nabla \mathbf{u}(x)|^2 dx &= \int_{B_r(x_0)} \text{Tr} \tilde{\sigma}_{ij}(x) dx \\ &= \int_{B_r(x_0)} \Delta \eta^\varepsilon - 2Q^\varepsilon \\ &\leq \int_{B_r(x_0)} \Delta \eta^\varepsilon \\ &= \int_{\partial B_r(x_0)} \nabla \eta^\varepsilon \cdot \nu \\ &\leq Cr \end{aligned}$$

with some tame constant  $C$  depending on the Lipschitz norms of  $\eta^\varepsilon$ , which is bounded by Lemmas 3.2 and 3.1. Now the result follows from Morrey's estimate.  $\square$

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